lectors in gaseous discharges," Phys. Rev. 28, 727-763 (1926).
[4] Medicus, G., "Diffusion and elastic collision losses of the 'fast electrons' in plasmas," J. Appl. Phys. 29, 903-908 (1958).
[5] Medicus, G., "Simple way to obtain the velocity distribution of the electrons in gas discharge plasmas from probe curves," J. Appl. Phys. 27, 1242-1248 (1956).
[6] Druyvesteyn, M., "Der Neidervoltbogen," Z. Physik 64, 781–798 (1930).
[7] Kerrisk, D. J., "Arc-type ion sources for electrical propulsion," TN 61-4, Aeronaut. Systems Div., Wright-Patterson Air Force Base (May 1961).
[8] Bohm, D., *Characteristics of Electrical Discharges in Magnetic Fields*, edited by A. Guthrie and R. K. Wakerling (McGraw Hill Book Co. Inc., New York, 1949), Chap. 3.

AUGUST 1963 AIAA JOURNAL VOL. 1, NO. 8

Concentrated Loads on Inflated Structures

LLOYD H. DONNELL*
Armour Research Foundation, Chicago, Ill.

The most efficient and practical mechanism for distributing concentrated loads into an inflated membranous envelope is a so-called "catenary curtain." This is a membranous strip attached to the envelope along one edge, whereas the other edge is scalloped, with the loads applied to the projecting points of the scallops. Such curtains have been used for transmitting the weight and other forces on the car of a nonrigid airship into the envelope, but these have been designed without consideration of the deformations involved. An analysis is presented of catenary curtains for applying normal or slightly oblique loads to an envelope. The relations between the forces and the changes in shape, displacements, and strains in the various elements of the curtain and the envelope are given, on the assumption that the curtain scallop spacing is small compared to the main dimensions of the envelope. The results are developed in the form of series that converge rapidly. They apply to relatively large displacements and unit strains up to the order of 0.1. Several examples are worked out.

Nomenclature

a,b,c = dimensions of catenary cable (Figs. 8 and 9)
b,f' = force per unit length between catenary curtain and envelope in direction of tie cable and normal to envelope surface
T,F = tension in catenary cable and its minimum value
0 = subscript indicating original value
p,p' = actual and "corrected" gas pressure in envelope, Eq. (7)
r,r' = radius of envelope before and after tensioning tie cable (Fig. 4)
u,u' = deflection of envelope in direction of tie cable and normal to envelope surface
W = tension in tie cable
x,y,s = coordinates of catenary cable perpendicular to, parallel to, and along cable
α,λ = angles between tie cable and envelope (Fig. 6)
β = $2b_0/a_0$
θ = $(b_1/b_0) - (1 + e_b)$
ϕ = $a(p'/F)^{1/2}$
$e_a,e_b,$
e_s,e_r = unit strains in a, b, s, and r

INFLATED structures, made of a tough membrane whose shape is maintained by a small internal pressure differential, have been used for many years. Because the pressure is small, they are insensitive to small leaks. They already have been used in space and are likely to be useful especially there because they make possible large structures of small weight, occupying little space before inflation.

All of the standard structural elements can be duplicated, for example, struts by inflated cylindrical tubes, but the chief application is in complete monocoque structures in which the membranous envelope provides both cover and structural stability. An important problem in connection with the design of such structures is that of applying concentrated loads to the envelope. Loads tangential to the envelope can be applied in a simple manner through patches, but these cause stress concentrations in the envelope which have been discussed by the author.[1, 2]

The most efficient and practical mechanism for distributing concentrated loads at any angle into the envelope is the so-called "catenary curtain." This is a membranous strip attached to the envelope along one edge, whereas the other edge is scalloped, with the loads applied through "tie-cables" to the projecting points of the scallops. The construction of a catenary curtain and the nomenclature that will be used are shown in Fig. 1.

Such curtains have been used for transmitting the weight and other forces on the car of a nonrigid airship into the envelope, but these have been designed without consideration of the deformations involved. This paper presents an analysis of catenary curtains for applying normal or slightly oblique loads to an envelope. It gives the relations between the forces and the changes in shape, displacements, and strains in the various elements of the curtain and envelope. This analysis can be used for calculating the change in shape produced by the elastic strains that occur when the structure is inflated initially, so as to allow for them in tailoring the curtain and the adjoining structure.

Since the membranous materials usually used are subject to plastic flow and creep, a more important application is likely to be to the problem of how the shape and load distribution will be affected by permanent strains due to yielding under overloads or to creep under normal loading over long periods of time. A study of the effect of permanent strains in a complete structure must take some account of the entire structure, but in such a study the catenary curtain presents special difficulties because of the complex relationship

Presented at the ARS 17th Annual Meeting and Space Flight Exposition, Los Angeles, Calif., November 13–18, 1962. This work was sponsored by General Development Corporation under a contract with the Bureau of Aeronautics, U. S. Navy.

* Staff Consultant; also Professor Emeritus of Mechanics, Illinois Institute of Technology, Chicago, Ill.

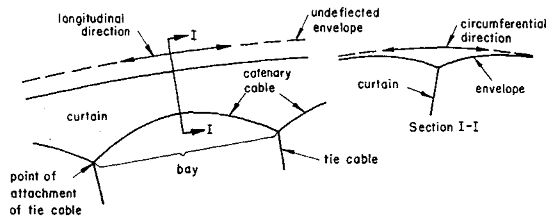

Fig. 1 Nomenclature used.

between the actions of its various elements. In the following theory, the shape of the curtain and the displacement of the point of attachment of the tie cable are calculated as a function of the tie cable tension, initially and after arbitrary strains have occurred in the elements of the curtain and the adjacent envelope. These relations can be incorporated in a more general study.

Besides the usual assumptions made in structural design, a number of approximate assumptions are made. To avoid almost insurmountable complications in the theory and its application to actual structures, the idealization is introduced that each bay is treated as if all the bays had the same shape, and the envelope is a cylinder with the curvature that it has at the bay studied. Thus, if the actual catenary curtain is as shown in Fig. 2, the theory for the tie cable attachment point Q is based upon the idealized envelope and curtain shown in Fig. 3. The errors that this idealization introduces largely average out and probably are unimportant insofar as overall studies of an inflated structure are concerned. The effect of the angles α and λ(Fig. 2), which the forces exerted by the tie cables and the tension in the curtain make with the "longitudinal" and "circumferential" directions defined in Fig. 1, may be important and are considered.

It is assumed that the force between the curtain and the envelope initially is distributed uniformly in the longitudinal direction, as seems obviously desirable, and that the curtain is under uniaxial tension in the direction of this force, as it would be if it is made slightly full in the perpendicular direction or is constructed of parallel threads or cords running in this direction, like the cord fabric used in making cord tires. (Such a curtain material is much more efficient than ordinary fabric in which the threads perpendicular to the tension not only are wasted but also weaken the threads that do the work, due to sawing action where the threads cross and to the angularities inherent in woven construction.)

A uniform uniaxial tension in the curtain requires that the "catenary" cable be of parabolic shape and have a nonuniform tension. Even if it is impractical to construct this cable with a correspondingly nonuniform cross section, the loss in its efficiency is more than made up by the efficiency of the curtain. Only the average strain over the bay in the catenary cable is considered strictly; the assumed distribution of strain along the cable is reasonable but not exact. The entire catenary curtain is assumed to have a constant longitudinal strain equal to the longitudinal strain in the envelope in this region, and there is assumed to be a unit change in the radius of the envelope equal to the circumferential strain in the envelope.

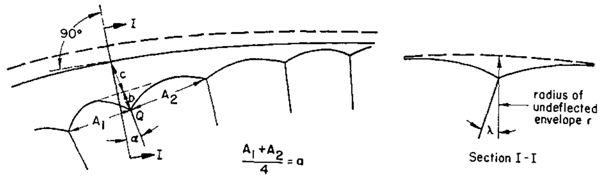

Fig. 2 Actual catenary curtain.

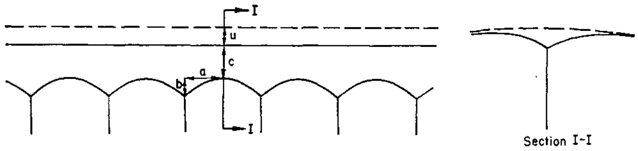

Fig. 3 Idealized catenary curtain.

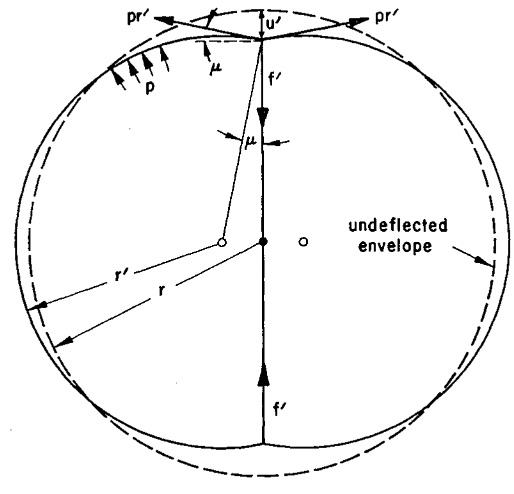

Fig. 4 Deflection of envelope by curtain.

The theory can take account accurately of any amount of strain, provided that this strain is nearly the same for all the elements, so that the geometric shape does not change greatly. Trial indicates that the theory is satisfactory if the maximum difference between the unit strains in the various elements does not exceed about 0.1.

The relation between normal deflection of the envelope and the normal force that is exerted upon it by the catenary curtain is taken as

$$u' = f'/2.8p \tag{1}$$

which is derived below. Since this relation does not involve the curvature of the envelope in the circumferential direction, it probably is not influenced much by any curvature in the longitudinal direction either. Although derived on the assumption of uniform conditions in the longitudinal direction, it probably is a fair approximation when they change gradually.

Figure 4 shows the cross section of a long cylindrical envelope of radius r before deformation. The full lines show the envelope deflected uniformly along its length by a diametral curtain. Each of the lobes into which the envelope now is divided will be cylindrical, with a new radius r'. Because of the internal pressure p, each lobe will have a circumferential tension per unit length pr'. Calling the tension per unit length in the curtain f', equilibrium of the joint between the lobes and the curtain in the vertical direction requires

$$f' = 2pr' \sin\mu \tag{2}$$

Assuming that the circumferential length of the envelope is

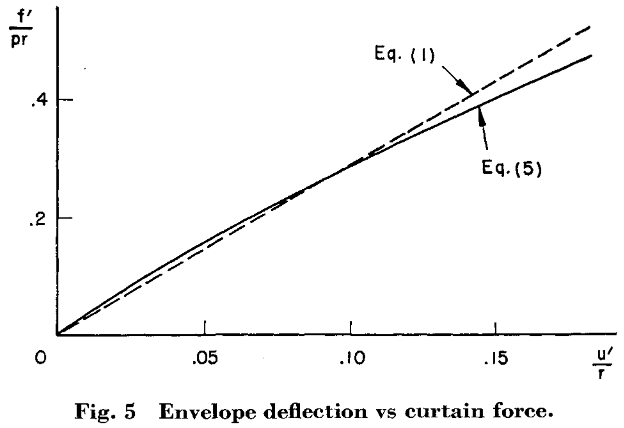

Fig. 5 Envelope deflection vs curtain force.

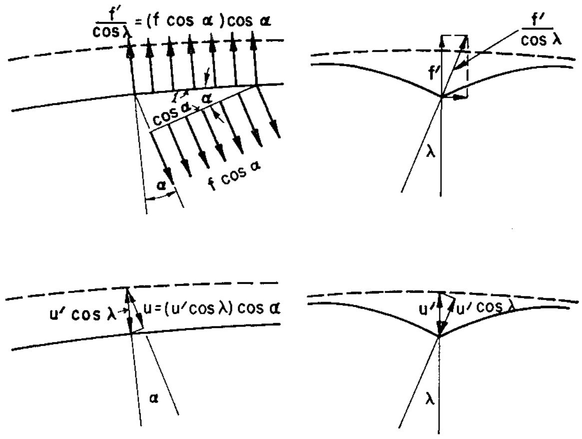

Fig. 6 Correction for angularity.

not changed, one has

$$2\pi r = (2\pi + 4\mu)r' \tag{3}$$

The deflection u' normal to the undeflected envelope is

$$u' = r - r' \cos\mu \tag{4}$$

Using Eq. (3) to eliminate r' from Eqs. (2) and (4),

$$\frac{f'}{pr} = \frac{2\pi \sin\mu}{\pi + 2\mu} \qquad \frac{u'}{r} = \frac{\pi(1 - \cos\mu) + 2\mu}{\pi + 2\mu} \tag{5}$$

A graphical solution of Eqs. (5) is given by finding f'/pr and u'/r for the same values of μ and then plotting the values so obtained against each other. This is done in Fig. 5, from which it will be seen that the linear relation given by Eq. (1) is a good approximation for values of u' up to more than $r/8$.

To take account of the angularity between the direction of f' and of the uniaxial tension in the curtain f and between u' and the deflection u in the direction of f, it can be seen from Fig. 6 that

$$\begin{aligned} f' &= f \cos^2\alpha \cos\lambda \\ u' &= u/(\cos\alpha \cos\lambda) \end{aligned} \tag{6}$$

Substituting these values in Eq. (1), one obtains

$$u = f \cos^3\alpha \cos^2\lambda/(2.8p) = f/p' \tag{7}$$

where

$$p' = 2.8p/(\cos^3\alpha \cos^2\lambda)$$

The equilibrium conditions of elements of the catenary cable now will be considered. The assumption that the catenary curtain is under uniaxial tension, constant in the direction of the tension, satisfies equilibrium and compatibility conditions in the curtain, if the resistance of the curtain material to strains perpendicular to the tension and to shear strains is negligible.

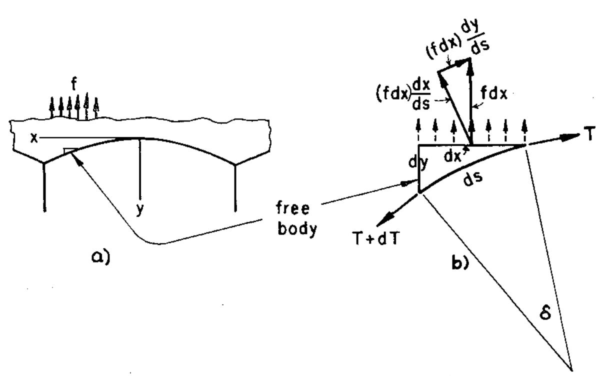

Fig. 7 Equilibrium of catenary cable element.

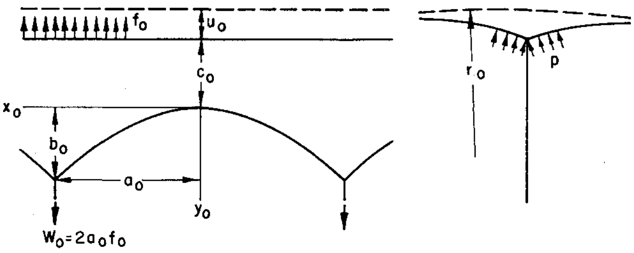

Fig. 8 Original bay shape and loading.

A free body such as shown in Fig. 7, consisting of an elemental length ds of the catenary cable and a triangular portion of the catenary curtain, is acted upon by the forces shown in Fig. 7b. The x, y axes are taken as shown in Fig. 7a, with the y axis in the direction of the tie cable and of the uniaxial tension f in the curtain. In general, both f and the tension T in the cable may be functions of x. Equilibrium of this free body in the directions parallel and perpendicular to ds requires that

$$(f\,dx)(dy/ds) = dT \tag{8}$$

$$(f\,dx)(dx/ds) = T\delta \tag{9}$$

The small angle δ is equal to ds divided by the radius of curvature of the cable curve. Using the mathematical expression for the radius of curvature, Eq. (9) becomes

$$(f\,dx)\frac{dx}{ds} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}\,T ds \tag{10}$$

Using $ds^2 = dx^2 + dy^2$, Eqs. (8) and (10) become

$$\frac{f\,dy/dx}{[1 + (dy/dx)^2]^{1/2}} = \frac{dT}{dx} \tag{11}$$

$$\frac{f[1 + (dy/dx)^2]^{1/2}}{d^2y/dx^2} = T \tag{12}$$

Eliminating T between Eqs. (11) and (12) and integrating once, one finds

$$f = F(d^2y/dx^2) \tag{13}$$

$$T = F[1 + (dy/dx)^2]^{1/2} \tag{14}$$

where F is a constant of integration which has the physical meaning of the minimum value of T.

The idealized catenary curtain in its original condition is shown in Fig. 8. From Eq. (13), using subscripts 0 to indicate the original values,

$$f_0 = F_0(d^2y_0/dx_0^2) \tag{15}$$

The condition that f_0 be constant requires that the curve be a parabola. In order to pass through the points 0, 0 and a_0, b_0, its equation must be

$$y_0 = b_0(x_0/a_0)^2 \tag{16}$$

From equilibrium of the whole catenary curtain, substituting Eq. (16) into (15),

$$W_0 = 2a_0f_0 \tag{17}$$

$$F_0 = (a_0^2/2b_0)f_0 = (a_0/4b_0)W_0 = (W_0/2\beta) \tag{18}$$

where $\beta = 2b_0/a_0$, and W_0 is the initial tie cable tension. The initial catenary cable tension is

$$T_0 = F_0\left[1 + \left(\frac{dy_0}{dx_0}\right)^2\right]^{1/2} = F_0\left(1 + \frac{4b_0^2}{a_0^4}x_0^2\right)^{1/2} = \frac{W_0}{2\beta}\left(1 + \beta^2\frac{x_0^2}{a_0^2}\right)^{1/2} \tag{19}$$

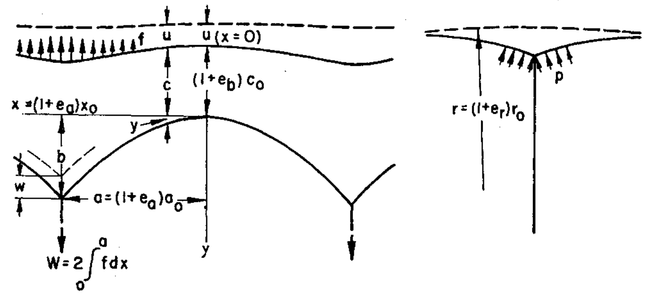

Fig. 9 Bay shape and loading after strain.

The maximum catenary cable tension is at the tie cable attachment point, $x_0 = a_0$:

$$T_{0(\max)} = (W_0/2\beta)(1+\beta^2)^{1/2} \tag{20}$$

From Eq. (7), the uniform deflection u_0 of the envelope is

$$u_0 = f_0/p' = W_0/(2a_0p') \tag{21}$$

The idealized catenary curtain in its final strained condition is shown in Fig. 9. The origin of coordinates is taken at the same point of the catenary cable as in the original condition, and the point on the cable whose original coordinates were x_0, y_0 now is assumed to have the coordinates x, y. As discussed previously, it is assumed that there has been a uniform unit tensile strain e_a in the x direction, so that $x = (1+e_a)x_0$, $a = (1+e_a)a_0$, and $x/a = x_0/a_0$. There also is assumed to have been a uniform strain e_b in the y direction, an average strain e_s in the catenary cable, and a circumferential strain and corresponding unit change in the radius of the envelope.

The shape of the catenary cable curve presumably no longer will be parabolic, and the uniaxial tension in the curtain, f, and the deflection u of the envelope will vary with x. The new cable curve can be represented by the power series:

$$y = \Sigma_n\, b_n(x/a)^{2n} \qquad n = 1, 2, 3, \ldots \tag{22}$$

where the b_n's are coefficients with the dimensions of length which are to be determined. The value of y when $x = a$ is

$$b = \Sigma_n b_n \tag{23}$$

From Eqs. (13, 14, and 7),

$$f = F(d^2y/dx^2) = (p'/\phi^2)\Sigma_n 2n(2n-1)b_n(x/a)^{2n-2} \tag{24}$$

where $\phi^2 = p'a^2/F$. Then

$$T_{(x=a)} = T_{\max} = (p'a^2/\phi^2)[1 + (\Sigma_n 2nb_n/a)^2]^{1/2} \tag{25}$$

$$u = f/p' = (1/\phi^2)\Sigma_n 2n(2n-1)b_n(x/a)^{2n-2} \tag{26}$$

$$u_{(x=0)} = 2b_1/\phi^2$$

In order for the curtain to have a uniform strain e_b in the y direction,

$$c + y = (1+e_b)(c_0 + y_0) \tag{27}$$

But from Fig. 9, it can be seen that

$$c = (1+e_b)c_0 + u_{(x=0)} - u \tag{28}$$

Now substitute Eq. (28) into (27), using (22, 26, and 16) and remembering that $x_0/a_0 = x/a$. Grouping the resulting terms in the form of a power series in x, one obtains

$$\left[b_1 - (1+e_b)b_0 - \frac{1}{\phi^2}\,3\cdot 4b_2\right]\left(\frac{x}{a}\right)^2 + \left[b_2 - \frac{1}{\phi^2}\,5\cdot 6b_3\right]\left(\frac{x}{a}\right)^4 \ldots + \left[b_{n-1} - \frac{1}{\phi^2}(2n-1)(2n)b_n\right]\left(\frac{x}{a}\right)^{2n-2} \ldots = 0 \tag{29}$$

In order for the left side of Eq. (29) to be zero for all values of x, each coefficient of the power series must separately be zero. Therefore, using the symbol $\theta = (b_1/b_0) - (1+e_b)$,

$$\begin{aligned} b_1/b_0 &= \theta + (1+e_b) \qquad b_2/b_0 = (\phi^2/3\cdot 4)\theta \\ b_3/b_0 &= (\phi^2/5\cdot 6)b_2/b_0 = (\phi^4/3\cdot 4\cdot 5\cdot 6)\theta \\ b_n/b_0 &= [2\phi^{2n-2}/(2n)!]\theta \qquad (n > 1) \end{aligned} \tag{30}$$

Using these values in Eq. (23),

$$\begin{aligned} b = \Sigma_n b_n &= (1+e_b)b_0 + (2b_0\theta/\phi^2)(\phi^2/2! + \phi^4/4! \ldots) \\ &= (1+e_b)b_0 + (2b_0\theta/\phi^2)(\cosh\phi - 1) \end{aligned} \tag{31}$$

Finally, consider the average strain e_s in the catenary cable. Figure 10 shows an element ds_0 of the cable in its original condition superposed upon the same element, now of length ds, as it is after the cable has been strained. Then

$$ds\cos\Delta - ds_0 = (dx - dx_0)(dx_0/ds_0) + (dy - dy_0)(dy_0/ds_0) \tag{32}$$

If the change in shape is only moderate, the angle Δ will be small, and one can take $\cos\Delta = 1$. Then the left side of Eq. (32) will represent the change in length of the element of the cable. Integrating this, one obtains the total change in length of half of the catenary cable, which is e_s times the original length of half of the cable, that is,

$$e_s\int_0^a \frac{ds_0}{dx}\,dx = \int_0^a \left[\frac{(dx - dx_0)}{dx}\frac{dx_0}{ds_0} + \frac{(dy - dy_0)}{dx}\frac{dy_0}{ds_0}\right]dx \tag{33}$$

Using Eqs. (16) and (22) for y_0 and y and remembering that $ds_0^2 = dx_0^2 + dy_0^2$, $x = (1+e_a)x_0$, and $a = (1+e_a)a_0$, this becomes

$$\frac{e_s}{\beta^2}\int_0^a \left[1 + \left(\frac{\beta x}{a}\right)^2\right]^{1/2} dx = \int_0^a \frac{e_a/\beta^2 - (x/a)^2 + \Sigma_n n(b_n/b_0)(x/a)^{2n}}{[1 + (\beta x/a)^2]^{1/2}}\,dx \tag{34}$$

Carrying out these integrations and solving for θ, one finds, using Eq. (30),

$$\theta = \frac{e_s - (2-B)e_a - (B-1)e_b}{(B-1)C_0 + B(\beta^2C_2 + \beta^4C_4 + \ldots \beta^nC_n \ldots)} \tag{35}$$

where

$$B = \frac{2(1+\beta^2)^{1/2}}{(1+\beta^2)^{1/2} + (\sinh^{-1}\beta)/\beta}$$

$$C_0 = 1 - \frac{(\phi/\beta)^2}{8} + \frac{(\phi/\beta)^4}{192} - \ldots (-1)^{q/2}.\frac{[3\cdot 5\ldots(q+1)](\phi/\beta)^q}{(4\cdot 6\ldots q)(q+2)!}$$

$$C_2 = \frac{1}{3} - \frac{C_0}{3}$$

$$C_4 = \frac{(\phi/\beta)^2}{30} - \frac{4C_2}{5}$$

$$C_6 = \frac{(\phi/\beta)^4}{840} - \frac{6C_4}{7}$$

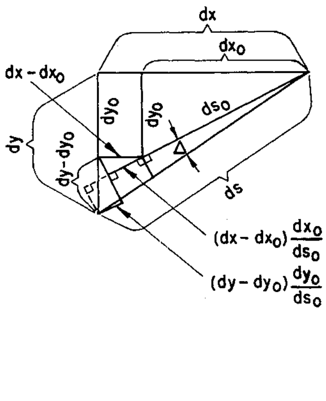

Fig. 10 Catenary cable strain.

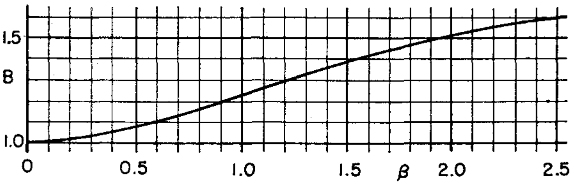

Fig. 11 B **as a function of** β**.**

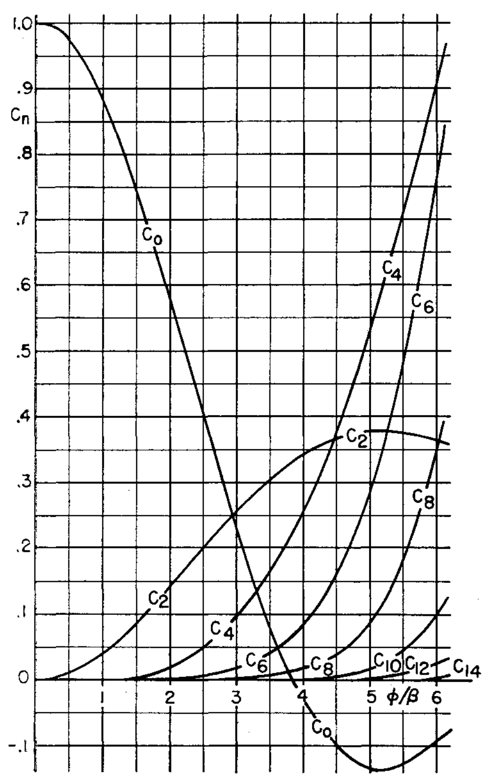

Fig. 12 C_n **as functions of** ϕ/β**.**

$$C_p = \frac{(\phi/\beta)^p}{(p+2)!} \ldots . (-1)^{(p+q)/2} \times$$
$$\frac{[(p+3)(p+5)\ldots(q+1)](\phi/\beta)^q}{[(p+2)(p+4)\ldots q]\,(q+2)!} \qquad (p, q = 4, 6 \ldots)$$

Values of B and of C_0, C_2 . . . are plotted in Figs. 11 and 12 as functions of β and ϕ/β.

From equilibrium of the entire catenary curtain in the direction of its tension, the tension W in each tie cable is, using Eqs. (24) and (30),

$$W = 2\int_0^a f dx = \frac{4F}{a}\,\Sigma_n n b_n = \frac{4p'a}{\phi^2}\,\Sigma_n n b_n$$
$$= \frac{4p'ab_0}{\phi^2}\left[1 + e_b + \frac{\theta}{\phi}\left(\phi + \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \frac{\phi^7}{7!}\ldots\right)\right]$$
$$= \frac{4(1+e_a)p'a_0b_0}{\phi^2}\left[1 + e_b + \theta\,\frac{(\sinh\phi)}{\phi}\right] \qquad (36)$$

Values of $\sinh\phi/\phi$ and of $\cosh\phi$, used in Eqs. (31, 36, and 37), are plotted in Fig. 13.

Finally, the displacement w, in the direction of W, of the point of attachment of the tie cable produced by strains e_a, e_b, e_s, and e_r is, from Figs. 8 and 9,

$$w = b + (1 + e_b)c_0 + u_{x=0} - b_0 - c_0 - u_0 - e_r r_0$$

which, by using Eqs. (21, 26, 30, and 31), can be put in the form

$$w = [c_0 + b_0 + (2b_0/\phi^2)]e_b - r_0 e_r + (2b_0/\phi^2)(1 + \theta\cosh\phi) - u_0 \qquad (37)$$

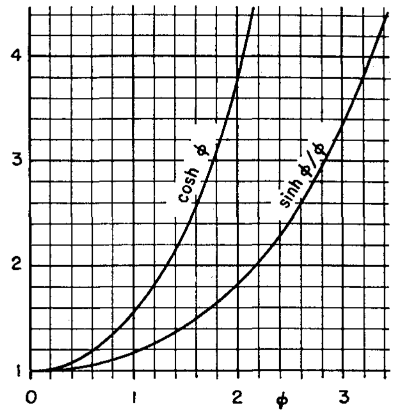

Fig. 13 Coshϕ **and (sinh**ϕ**)/**ϕ **as function of** ϕ**.**

where $u_0 = W_0/(2a_0p')$.

Application of Theory

In a given application, the following will be known: a_0, b_0, c_0, r_0 (the original values of the dimensions defined in Fig. 2), $p' = 2.8\,p/(\cos^3\alpha\,\cos^2\lambda)$ (where p is the envelope pressure and the angles α, λ are defined in Fig. 2), W_0 (the original load in the tie cable being studied), and the strains e_a, e_r in the envelope in the longitudinal and circumferential directions, e_b in the catenary curtain in the direction of its tension, and e_s in the catenary cable.

The load vs displacement relation at the point of attachment of the tie cable, after the strains have occurred, then can be obtained as follows. Values of ϕ are assumed; the range of values of interest probably will lie between $\phi = (b_0/u_0)^{1/2}$ and $\phi = 2(b_0/u_0)^{1/2}$, where $u_0 = W_0/(2a_0p')$ is the original deflection of the envelope. The values of $\beta = 2b_0/a_0$ and of ϕ/β now can be calculated and from these B and C_0, C_2 . . . found from Figs. 11 and 12. The value of θ then is given by Eq. (35), after which W and w can be found from Eqs. (36) and (37), and a W vs w curve can be plotted. The new shape of the catenary curtain is given by $a = (1 + e_a)a_0$, $b = (1 + e_b)b_0$, and $y = \Sigma_n b_n(x/a)^{2n}$, where the b_n's are given by Eq. (30). Stresses, before and after straining, can be calculated from Eqs. (17, 20, 24, and 25). The series converge rapidly. Some examples of application of the theory are given below.

1) All strains being zero, $e_a = e_b = e_s = e_r = 0$. From Eq. (35), $\theta = 0$, and from Eqs. (36) and (37), $W = 4p'a_0b_0/\phi^2$ and $w = 2b_0/\phi^2 - W_0/(2a_0p')$. Values of W and w can be calculated for the same values of ϕ and plotted together to give a load vs displacement curve. For $W = W_0$, $\phi^2 = 4p'a_0b_0/W_0$ and $w = 0$ as expected.

2) All strains being equal, $e_a = e_b = e_s = e_r = e$. From Eq. (35), $\theta = 0$, and from Eq. (36), $W = 4(1 + e)^2p'a_0b_0/\phi^2$. For $W = W_0$, $\phi^2 = 4(1 + e)^2p'a_0b_0/W_0$, and from Eq. (37), $w = (b_0 + c_0 - r_0)e - u_0e/(1 + e)$. It is interesting to note that the deflection of the envelope after straining, $W_0/(2ap')$, is smaller than the deflection before straining, $W_0/(2a_0p')$.

3) All strains but one being equal to zero, let $e_a = e_b = e_r = 0$ and $e_s = 0.1$. Assume that $b_0 = c_0 = u_0 = a_0/2$. Then $\beta = 2b_0/a_0 = 1$, and, from Fig. 11, $B = 1.23$. Assume values of ϕ of 1, 1.5, and 2, as suggested in the foregoing. Then from Fig. 12 and Eq. (35), one finds

$$\begin{aligned}
\phi &= \phi/\beta = 1, 1.5, 2\\
C_0 &= 0.880, 0.742, 0.577\\
C_2 &= 0.041, 0.087, 0.141\\
C_4 &= 0.000, 0.006, 0.020\\
C_6 &= 0.000, 0.000, 0.002\\
\theta &= 0.397, 0.347, 0.302
\end{aligned}$$

From Eqs. (21) and (36),

$$\frac{W}{W_0} = \frac{2b_0}{\phi^2 u_0}\left(1 + \theta\,\frac{\sinh\phi}{\phi}\right) = 2.94, 1.32, 0.785$$

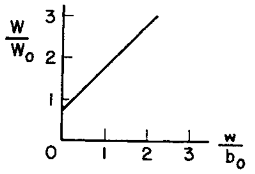

Fig. 14 Load-displacement of point of attachment after 10% cable strain.

$$\frac{w}{b_0} = \frac{2}{\phi^2}(1 + \theta \cosh\phi) - 1 = 2.22, 0.61, 0.07$$

Figure 14 shows the generalized load vs displacement curve obtained by plotting corresponding values of W/W_0 vs w/b_0. Figure 15 shows the original shape of the catenary curtain and its shape after straining, for the three loading conditions calculated. These curves are plotted from the relations

$$\frac{y_0}{b_0} = \left(\frac{x_0}{a_0}\right)^2 = \left(\frac{x}{a}\right)^2$$

$$\frac{y}{b_0} = (\theta + 1)\left(\frac{x}{a}\right)^2 + \frac{\phi^2\theta}{12}\left(\frac{x}{a}\right)^4 + \frac{\phi^4\theta}{360}\left(\frac{x}{a}\right)^6 \cdots$$

$$\frac{u_0}{b_0} = 1$$

$$\frac{u_{(x=0)}}{b_0} = \frac{2}{\phi^2}\frac{b_1}{b_0} = 2.79, 1.19, 0.66$$

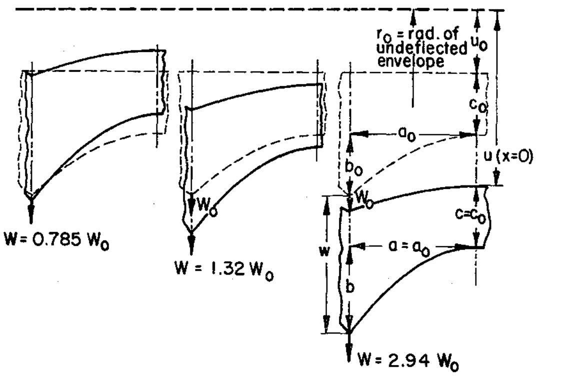

Fig. 15 Change in shape due to 10% cable strain.

References

[1] Donnell, L. H., "Stress concentration at the ends of long reinforcements," Natl. Geograph. Soc. Stratosphere Ser. **2**, 245 (1936).

[2] Donnell, L. H., "Stress concentrations due to elliptical discontinuities in plates under edge forces," *Theodore von Kármán Anniversary Volume* (California Institute of Technology, Pasadena, Calif., 1941), p. 293.

AUGUST 1963 AIAA JOURNAL VOL. 1, NO. 8

Case-Bounded Elastic-Plastic and Nonlinear Elastic Hollow Cylinders

M. Bieniek,* M. Shinozuka,† and A. M. Freudenthal‡
Columbia University, New York, N. Y.

An elastically case-bonded hollow cylinder of infinite length, the mechanical response of which is perfectly elastic-plastic or nonlinearly elastic, is considered. For a perfectly elastic-plastic cylinder with the Tresca yield condition and its associated flow rule, W. T. Koiter's solution is extended to the problem of the cylinder contained in an elastic shell; the effect of the shell on the stresses is demonstrated with numerical examples. The same problem is solved for a nonlinear elastic cylinder, the second invariants J and I of the stress and strain deviator of which are assumed to have a relation $J = 4(G - gI)_2 I$, G and g being material constants; incompressibility is introduced to make the analysis simple. The critical state is defined in such a way that failure occurs at the point where I reaches a critical value I_{cr}. Comparison of the stresses in the elastic-plastic and in the nonlinear elastic cylinder for the inner surface of the cylinder reaching the critical condition shows little difference, at least for the specific values of parameters chosen for computation.

1. Introduction

THE present investigation deals with an ideally elastic-plastic and a nonlinearly elastic hollow cylinder of infinite length enclosed in an elastic shell and subject to an internal pressure. The nonlinearities assumed are intended to reflect approximately the mechanical response of the solid propellant material.

In the first part, the plane strain problem of an elastically compressible and plastically incompressible elastic-plastic hollow cylinder contained in an elastic shell subject to an internal pressure is considered. Little attention has been paid to this problem so far, although the same problem without elastic shell has been solved under various yield conditions and stress-strain relations.

For a perfectly plastic material, the component ϵ_{ij} of the (total) strain can be written as the sum of the components of the elastic and the plastic strains, ϵ_{ij}^E and ϵ_{ij}^P:

$$\epsilon_{ij} = \epsilon_{ij}^E + \epsilon_{ij}^P \tag{1}$$

The incompressibility of the plastic deformation

$$\epsilon_{kk}^P = 0 \tag{2}$$

is assumed; the usual convention of summation over repeated subscripts is adopted.

Received February 18, 1963. This research was supported by the Office of Naval Research under Contract Nonr 266(78).

* Associate Professor of Civil Engineering. Member AIAA.

† Assistant Professor of Civil Engineering.

‡ Professor of Civil Engineering. Member AIAA.